A Measure of Interaction For Binary Targets

Raymond Anderson
rayanrisk@gmail.com

Presented at
Credit Scoring and Control XVII
August 27th, 2021

Purpose

- Aid understanding of relationships within data
- Present interaction matrix
- Promote use of simpler and interpretable models
- Identify segments requiring different treatment
- Assess whether interactions are addressed by a model(s)
- Measure changes in interactions over time

Art history

Closed form

Open form

Heinrich Wölfflin (1915): *Kunstgeschichtliche Grundbegriffe* ("Principles of Art History").

Statistics

Closed form

Can be achieved using in a finite number of steps using standard operators.

A formula the provides an exact answer given a finite amount of data.

Open form

Requires a large number of steps and/or non-standard operators.

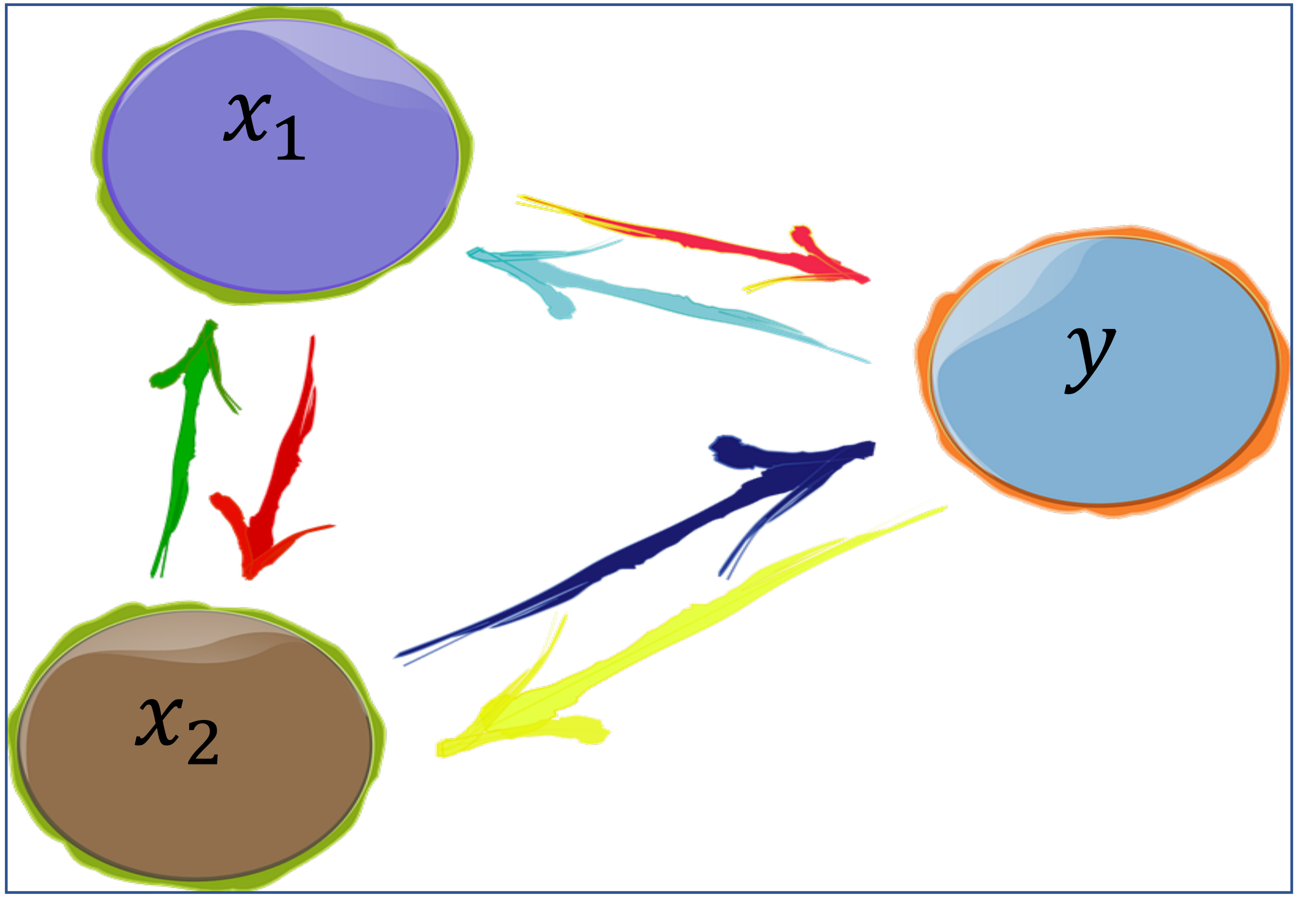
An equation that needs to be solved, often inexactly or with multiple solutions.

Closed forms are possible for open form expressions, if the resulting value is exact—e.g. for a perpetuity the open form is

$$PV = v/(1+i) + v/(1+i)^2 + v/(1+i)^3 + \cdots + v/(1+i)^\infty$$

but closed form is $PV = v/i$

Interaction—definition



The change in
the effect of x_1
on y as x_2 varies.

A problem for
linear models.
MODEL RISK!

Varying approaches at measurement

Interaction measure ($\mathfrak{I}$)	$\mathcal{F}$ Functional	$\circ$ Operation	I Identity of $\circ$	$\triangle$ Operation	Applications
Covariance	Expectation (E)	Multiplication	1	Multiplication	**Neuroscience** (Kohn & Smith, 2005; Staude et al., 2010); **gene epistasis, economics** (Miller, 2013); **signal processing** (Sahidullah & Kinnunen, 2016); **risk analysis and management** (Cox L. A. Jr., 2009)
Mutual information	Negative entropy ($-H$)	Addition	0	Addition	**Gene epistasis** (Margolin et al., 2006; Moore et al., 2006); **machine learning** (Jakulin & Bratko, 2003; Jakulin, 2005); **astronomy** (Pandey & Sarkar, 2017)
Additive model or ANOVA	Response	Addition	0	Addition	**Stressor interactions** (Chen et al., 2008); **gene epistasis** (Matsui and Ehrenreich, 2016); **food webs** (O'gorman & Emmerson, 2009); **political studies** (Sigal et al., 1988)
Multiplicative model or Bliss Independence	Survival or fitness	Addition	0	Multiplication	**Multiple predator studies** (Sih et al., 1998); **drug interactions** (Bliss, 1939; Yeh et al., 2006); **gene epistasis** (Segrè et al., 2005)

Tekin, Elif; Yeh, Pamela J. & Savage, Van M. [2018-10-23] "General Form for Interaction Measures and Framework for Deriving Higher-Order Emergent Effects". *Frontiers in Ecology and Evolution*, doi 10.3389/fevo.2018.0166.

Some references

- **Karaca-Mandic, Pinar & Norton, Edward [2011-08]**. "Interaction Terms in Non-Linear Models". *Health Services Research.*
- **Preacher, Kristopher J [2021]** *A primer on interaction effects in multiple linear regression.* Vanderbilt University. www.quantpsy.org/interact/interactions.htm
- **Tekin, Elif; Yeh, Pamela J. & Savage, Van M. [2018-10-23]** "General Form for Interaction Measures and Framework for Deriving Higher-Order Emergent Effects". *Frontiers in Ecology and Evolution*, doi 10.3389/fevo.2018.0166.
- **Xia, Yin; Cai, Tianxi; Cai, T Tony [2018]** "Two-Sample Tests for High-Dimensional Linear Regression with an Application to Detecting Interactions". Statistics Sinica 28:63-92. https://doi.org/10.5705/ss.202016.0032
- **Finsaas, Megan C & Goldstein, Brandon L [2021]** "Do Simple Slopes Follow-Up Tests Lead Us Astray"? Advancements in the Visualisation and Reporting of Interactions". Psychological Methods 26(1):38–60. https://doi.org/10.1037/met0000266
- **Oishi, Takuya; Hyashi, Yoshihiro; Noguchi, Miho; Yano, Fumiaki; Kumada, Shungo; Takayama, Kozo, Okada, Kotaro; Onuki, Yoshinori [2020]** "Creation of novel large dataset comprising several granulation methods and the prediction of tablet properties from critical material attributes and critical process parameters using regularized linear regression models including interaction terms". *International Journal of Pharmaceutics* 577:119083.
- **Buza, Krisztian; Peška, Ladislav; Koller, Júlia [2020]** "Modified linear regression predicts drug-target interactions accurately." *PLoS ONE* 15(4):e0230726. https://doi.org/10.1371/journal.pone.0230726

What do we want

1. Closed form
2. Can set benchmarks, OR
   Use for hypothesis tests
3. Implementable
4. Can provide a pairwise (or higher) "interaction matrix"

Two-way Anova

Closed form for continuous variables

Laboratory	Material 1	Material 2	
1	8,12,12,9,6 $\bar{X}=9.4$	3,8,4,6,1 $\bar{X}=4.4$	$\bar{X}=6.9$
2	5,8,4,2,6 $\bar{X}=5$	4,6,3,2,1 $\bar{X}=3.2$	$\bar{X}=4.1$
	$\bar{X}=7.2$	$\bar{X}=3.8$	$\bar{\bar{X}}=5.5$

r=5, a=2,b=2

$SS_{total}=\sum_{k=1}^{r}\sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ijk}-\bar{X}_{...})^2=\Sigma(X-\bar{\bar{X}})^2=(8-5.5)^2+(12-5.5)^2+(12-5.5)^2+(9-5.5)^2+(6-5.5)^2+(3-5.5)^2+(8-5.5)^2+(4-5.5)^2+(6-5.5)^2+(1-5.5)^2+(5-5.5)^2+(8-5.5)^2+(4-5.5)^2+(2-5.5)^2+(6-5.5)^2+(4-5.5)^2+(6-5.5)^2+(3-5.5)^2+(2-5.5)^2+(1-5.5)^2=201$

$SS_{within}=\sum_{k=1}^{r}\sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ijk}-\bar{X}_{ij.})^2=\Sigma(X-\bar{X}cell)^2=(8-9.4)^2+(12-9.4)^2+(12-9.4)^2+(9-9.4)^2+(6-9.4)^2+(3-4.4)^2+(8-4.4)^2+(4-4.4)^2+(6-4.4)^2+(1-4.4)^2+(5-5)^2+(8-5)^2+(4-5)^2+(2-5)^2+(6-5)^2+(4-3.2)^2+(6-3.2)^2+(3-3.2)^2+(2-3.2)^2+(1-3.2)^2=91.2$

$SS_{Lab\ (row\ factor)}=rb\sum_{i=1}^{a}(\bar{X}_{i..}-\bar{X}_{..})^2=5*2\{(6.9-5.5)^2+(4.1-5.5)^2\}=39.2$

$SS_{Materiall\ (column\ factor)}=ra\sum_{j=1}^{b}(\bar{X}_{.j.}-\bar{X}_{..})^2=5*2\{(7.2-5.5)^2+(3.8-5.5)^2\}=57.8$

$SS_{LabXMaterial(Interaction)}=SS_t-SS_W-SS_A-SS_B=12.8$

$$M1 = M2\ ?$$
$$L1 = L2\ ?$$
$$M \perp L\ \ ?$$

Uses F-test

Hosmer-Lemeshow

Closed form for binary

Originally designed to assess goodness of fit of logistic-regression models

$$HL = \sum_{i=1}^{|A|}\sum_{j=1}^{|B|}\left(\frac{\left(\bar{\bar{S}}_{ij} - \hat{S}_{ij}\right)^2}{\hat{S}_{ij}} + \frac{\left(\bar{\bar{F}}_{ij} - \hat{F}_{ij}\right)^2}{\hat{F}_{ij}}\right) \approx \chi^2_{d.f.=|A|+|B|-2}$$

$$= \sum_{i=1}^{|A|}\sum_{j=1}^{|B|}\left(\frac{\left(\bar{\bar{N}}_{ij} \times \left(\bar{\bar{p}}_{ij} - \hat{p}_{ij}\right)\right)^2}{\bar{\bar{N}}_{ij} \times \hat{p}_{ij} \times \left(1 - \hat{p}_{ij}\right)}\right)$$

Where:
S—Success, F—Failure
N—Total
p—rate/probability
$\bar{\bar{\ }}$ —known (double bar)
$\hat{\ }$ —estimated (hat)
A—first dimension
B—second dimension
$|\ |$—class count

Assumes equally-sized quantiles
Varies greatly with population size N
Measures interaction if $\sum N\bar{\bar{p}} = \sum N\hat{p}$

Open form—differences in semi-elasticities (DIS)

Basic regression

$$f(Y_A) = \alpha_A + \hat{\beta}_A x_A + \varepsilon_A$$
$$f(Y_B) = \alpha_B + \hat{\beta}_B x_B + \varepsilon_B$$

Tests?

$$H_0: \hat{\beta}_A = \hat{\beta}_B$$
$$H_1: \hat{\beta}_A \neq \hat{\beta}_B$$

Xia, Yin; Cai, Tianxi; Cai, T Tony [2018] "Two-Sample Tests for High-Dimensional Linear Regression with an Application to Detecting Interactions". Statistics Sinica 28:63-92. https://doi.org/10.5705/ss.202016.0032

Open form—interaction terms

Basic regression

$$f(Y) = \alpha_A + \hat{\beta}_{A1}x_1 + \hat{\beta}_{A2}x_2 + \varepsilon_A$$

Interaction term

$$f(Y) = \alpha_B + \hat{\beta}_{B1}x_1 + \hat{\beta}_{B2}x_2 + \boldsymbol{\beta_{B12} f(x_1, x_2)} + \varepsilon_B$$

Tests?

$$H_0: \quad \beta_{B12} = 0$$
$$H_1: \quad \beta_{B12} \neq 0$$

Logistique

Probability estimate

$$\hat{p} = f(\boldsymbol{x})$$

ERROR! - Log-likelihood residual

$$\hat{\ell}_i = \overline{\overline{w}}_i \times -\begin{bmatrix} \ln(\hat{p}_i) & | \text{ "F"} \\ \ln(1-\hat{p}_i) & | \text{ "S"} \end{bmatrix}$$

Where: $\overline{\overline{w}}$ – weight; F & S – Failure versus Success

Average LLR

$$\hat{\ell}_\Sigma = \sum \hat{\ell}_i \Big/ \sum \overline{\overline{w}}_i$$

Naïve log-likelihood

$$\acute{\ell}_g = -\begin{bmatrix} p_g \times \ln(p_g) + (1-p_g) \times \ln(1-p_g) & \mid 0 < p_g < 1 \\ 0 & \mid p_g \text{ in } [0,1] \end{bmatrix}$$

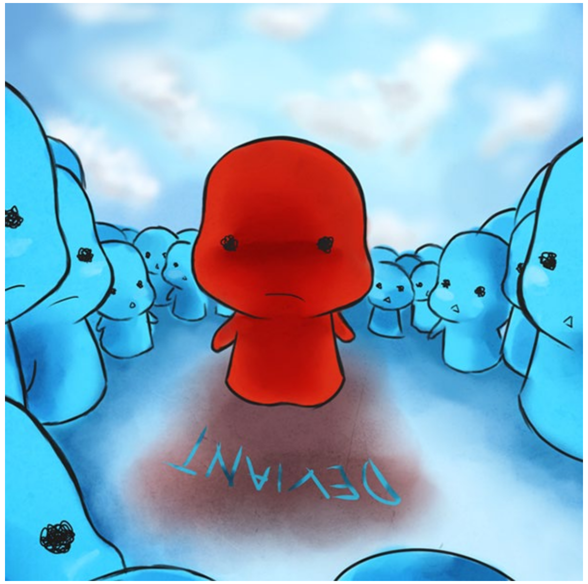
Where: p_g —may be an estimated or known rate for a group, $\hat{p}$ or $\bar{\bar{p}}$

Known versus estimated naïveté

$$\bar{\bar{\acute{\ell}}}_g = -\left(\bar{\bar{p}}_g \times \ln\left(\boxed{\bar{\bar{p}}_g}\right) + (1-\bar{\bar{p}}_g) \times \ln\left(1-\boxed{\bar{\bar{p}}_g}\right)\right)$$

$$\hat{\acute{\ell}}_g = -\left(\bar{\bar{p}}_g \times \ln\left(\boxed{\hat{p}_g}\right) + (1-\bar{\bar{p}}_g) \times \ln\left(1-\boxed{\hat{p}_g}\right)\right)$$

Likelihood and deviance



Likelihood

$$\mathfrak{L}_g = \exp(\ell_g)$$

Deviance

$$D_g = \mathfrak{L}_g^2 = \exp(\ell_g \times 2)$$

Likelihood Ratio

$$\mathbb{L} = \mathfrak{L}_0 / \mathfrak{L}_1$$

Basis for model comparison
and variable selection

Power and Inaccuracy

Power $$\odot = \left(\widehat{\ddot{D}} - \widehat{\acute{D}}\right) / \left(1 - \widehat{\acute{D}}\right)$$

Informed versus **naïve** for estimate

Inaccuracy $$\otimes = \left(\widehat{\acute{D}} - \bar{\bar{\acute{D}}}\right) / \bar{\bar{\acute{D}}}$$

Naïve **estimate** versus naïve **rate**

Source: unknown????

Interaction

Model inaccuracy $$\otimes_{\Sigma} = \left(\hat{\acute{D}}_{\Sigma} - \bar{\bar{\acute{D}}}_{\Sigma}\right) / \bar{\bar{\acute{D}}}_{\Sigma}$$

Pairwise inaccuracy
(can be done for higher orders)

$$\otimes_{A,B} = \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} \otimes_{i,j} \times f_{i,j}$$

Where: $f_{i,j} = n_{i,j} / n_{\Sigma}$

Interaction measure $$\boxtimes_{A,B} = \otimes_{A,B} - \otimes_{\Sigma}$$

$\boldsymbol{f_{i,j}^{2}}$**?**

Estimations—Appendices

ℵ=Natural log of odds

Σ =Full population

Naïve prediction $\hat{\aleph}_g = \ln(\bar{\bar{p}}_\Sigma/(1-\bar{\bar{p}}_\Sigma)) = \ln(\bar{\bar{p}}_\Sigma)$

Naïve Bayes $\hat{\aleph}_g = \bar{\bar{\aleph}}_\Sigma + \sum_{i=X,Y,Z} (\bar{\bar{\aleph}}_i - \bar{\bar{\aleph}}_\Sigma)$

X/Z interaction term $\hat{\aleph}_g = \bar{\bar{\aleph}}_{X,Z}$

Regressed $\hat{\aleph}_g = \bar{\bar{\aleph}}_\Sigma + .5472\mathcal{W}_{X,Y} + .9542\mathcal{W}_{X,Z} + .0809\mathcal{W}_{Y,Z}$

Problem!

Results vary greatly for different levels of p
It is not possible to set global benchmarks
Highest values are where $\bar{\bar{p}}$ is 50 percent

Solution…

Normalise all values to assume $\overline{\overline{p}}$ is 50 percent,
i.e. adjust weights, rates, and estimates

Results vary with overall risk

i	1	2	3	4	5	6	7	8	9	Total	Average
w	1	10	10	10	1	1	10	10	10	63	
"*F*"	1	0	0	0	1	1	0	0	0	3	4.762%
p Est	12.0%	2.0%	3.0%	8.0%	15.0%	12.0%	5.0%	3.0%	7.0%		
n Est	0.12	0.20	0.30	0.80	0.15	0.12	0.50	0.30	0.70	3.19	5.063%
ll	2.1203	0.0202	0.0305	0.0834	1.8971	2.1203	0.0513	0.0305	0.0726		
*w * ll*	2.1203	0.2020	0.3046	0.8338	1.8971	2.1203	0.5129	0.3046	0.7257	9.0213	0.1432

Label	Rate	Odds	Log-L'hood	Like-lihood	Devi-ance	Inaccuracy
Naive Obs	4.762%	20.000	0.1914	1.2110	1.4665	0.019%
Naive Est	5.063%	18.749	0.1915	1.2111	1.4668	**Power**
Test Est			0.1432	1.1540	1.3316	28.960%

Weight of 0.5 provides 1/1 odds, results in highest possible values

Weight	Power
0.001	16.1%
0.01	23.9%
0.1	40.9%
0.5	50.4%
1	48.4%
10	**29.0%**
100	17.7%
1000	12.8%

A balancing act

Record-level

Similar to weight of evidence

Estimate $\hat{\mathfrak{p}}_i = 1/\left(1 + exp\left(\widehat{\aleph}_i - \bar{\bar{\aleph}}_\Sigma\right)\right)$

where:

$\widehat{\aleph}_i$ —natural log of odds for the record−level estimate;

$\bar{\bar{\aleph}}_\Sigma$ —ditto, for the known population

Weight $\bar{\bar{\mathfrak{w}}}_i = \bar{\bar{w}}_i \times \left(1 + \bar{\bar{\theta}}_\Sigma\right) \Big/ \left(2 \times \begin{bmatrix} 1|"F" \\ \bar{\bar{\theta}}_\Sigma|"S" \end{bmatrix}\right)$

where: $\bar{\bar{\theta}}_\Sigma$ −known population odds

Group balancing

Rate $\bar{\bar{\mathfrak{p}}}_g = 1/\left(1 + exp\left(\boxed{\bar{\bar{\aleph}}_g} - \bar{\bar{\aleph}}_\Sigma\right)\right)$

Estimate $\hat{\mathfrak{p}}_g = 1/\left(1 + exp\left(\boxed{\hat{\aleph}_g} - \bar{\bar{\aleph}}_\Sigma\right)\right)$

where: $\bar{\bar{\aleph}}_g$—natural log of odds for a group's known rate;
$\bar{\bar{\aleph}}_\Sigma$—ditto, for a groups' overall estimate

% of total

$$\bar{\bar{\mathfrak{f}}}_g = \frac{\bar{\bar{f}}_g}{2} \times \left(\begin{array}{c} \bar{\bar{p}}_g \times \left(1 + \bar{\bar{\theta}}_\Sigma\right) + \\ \left(1 - \bar{\bar{p}}_g\right) \times \left(1 + \bar{\bar{\theta}}_\Sigma\right) / \bar{\bar{\theta}}_\Sigma \end{array} \right)$$

where: $\bar{\bar{\theta}}_\Sigma$ —known population odds

Balanced Naive

$$\boxtimes_{\Sigma}=\sum_{i=1}^{|A|}\sum_{j=1}^{|B|}\left(4-\bar{\bar{\acute{\mathfrak{D}}}}_{i,j}\right)/\bar{\bar{\acute{\mathfrak{D}}}}_{i,j}\times\mathfrak{f}_{i,j}$$

Can apply without estimates

Fragmentation correction?

$$\acute{\boxtimes}_{\Sigma}=\boxtimes_{\Sigma}\times\sum_{i=1}^{|A|}\sum_{j=1}^{|B|}\mathfrak{f}_{i,j}{}^{2}$$

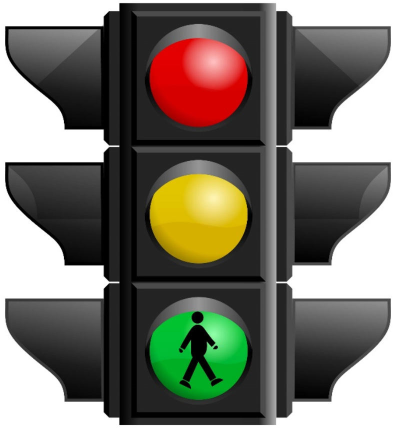
Issues!!!!

Values increase with fragmentation

Benchmarks may vary with circumstances

Not able to do hypothesis testing

Traffic lights?



No model	With model
10% or more?	2.0% or more?
2 to 10%?	0.2% to 2.0%?
Less than 2%?	Less than 0.2%?

Summary

Few interaction measures exists

Closed-form measure for binary targets presented

Based on log-likelihood residuals/inaccuracies

Can calculate naïve measures, and assess models' ability to address interactions

Future research

What benchmarks should be used, if any?

Can it be normalized for fragmentation?

Can it be used for hypothesis testing?

What are the alternative measures/approaches?

Appendices

All we know!

X	1	2	3	1	2	3	1	2	3	1	2	3	
Y	1	1	1	2	2	2	1	1	1	2	2	2	
Z	1	1	1	1	1	1	2	2	2	2	2	2	Total
Count	1 000	2 000	1 200	4 000	1 000	600	750	1 500	900	3 000	750	450	17 150
% of Total	5.83%	11.66%	7.00%	23.32%	5.83%	3.50%	4.37%	8.75%	5.25%	17.49%	4.37%	2.62%	100.00%
Failures	48	182	200	129	63	100	158	176	56	632	68	17	1 829
Rate	4.76%	9.09%	16.67%	3.23%	6.25%	16.67%	21.05%	11.76%	6.25%	21.05%	9.09%	3.85%	10.7%
Odds	**20.00**	**10.00**	**5.00**	**30.00**	**15.00**	**5.00**	**3.75**	**7.50**	**15.00**	**3.75**	**10.00**	**25.00**	8.38
Log Odds	3.00	2.30	1.61	3.40	2.71	1.61	1.32	2.01	2.71	1.32	2.30	3.22	2.13
WOE	0.870	0.177	-0.516	1.276	0.582	-0.516	-0.804	-0.111	0.582	-0.804	0.177	1.093	0.000

Perfect interaction between X and Z

	X1	X2	X3	Y1	Y2	Z1	Z2
Count	8 750	5 250	3 150	7 350	9 800	9 800	7 350
% of Total	51.02%	30.61%	18.37%	42.86%	57.14%	57.14%	42.86%
Failures	966	489	374	820	1 009	721	1 108
Rate	11.04%	9.31%	11.86%	11.16%	10.29%	7.36%	15.07%
Odds	8.06	9.74	7.43	7.96	8.72	12.59	5.64
Log Odds	2.09	2.28	2.01	2.07	2.17	2.53	1.73
WOE	-0.039	0.150	-0.120	-0.051	0.040	0.407	-0.397

Naïve prediction

$$\hat{\aleph}_g = \ln(\hat{p}_\Sigma/\neg\,\hat{p}_\Sigma) = \ln(\bar{\bar{p}}_\Sigma)$$

		Total			Fail %			Estimate %			RESULTS: Power		
		Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot
Inputs	X1	5 000	3 750	8 750	3.53%	21.05%	11.04%	**10.66%**	**10.66%**	**10.66%**	0.00%	0.00%	0.00%
	X2	3 000	2 250	5 250	8.14%	10.87%	9.31%	**10.66%**	**10.66%**	**10.66%**	0.00%	0.00%	0.00%
	X3	1 800	1 350	3 150	16.67%	5.45%	11.86%	**10.66%**	**10.66%**	**10.66%**	0.00%	0.00%	0.00%
	Tot	9 800	7 350	17 150	7.36%	15.07%	10.66%	**10.66%**	**10.66%**	**10.66%**	0.00%	0.00%	0.00%
Log likelihd		Naive Obs			Naïve Est			Estimate			Inaccuracy		
	X1	0.153	0.515	0.347	0.188	0.560	0.347	**0.188**	**0.560**	**0.347**	7.3%	9.6%	0.0%
	X2	0.282	0.344	0.310	0.286	0.344	0.311	**0.286**	**0.344**	**0.311**	0.7%	0.0%	0.2%
	X3	0.451	0.212	0.364	0.467	0.229	0.365	**0.467**	**0.229**	**0.365**	3.3%	3.5%	0.1%
	Tot	0.263	0.424	0.339	0.269	0.433	0.339	**0.269**	**0.433**	**0.339**	1.3%	1.9%	0.0%
Deviance		Naive Obs			Naïve Est			Estimate			Interaction		
	X1	1.357	2.799	2.003	1.456	3.066	2.004	1.456	3.066	2.004	2.1%	2.1%	4.2%
	X2	1.759	1.989	1.858	1.771	1.989	1.862	1.771	1.989	1.862	0.1%	0.0%	0.1%
	X3	2.462	1.527	2.071	2.545	1.580	2.074	2.545	1.580	2.074	0.4%	0.3%	0.6%
	Tot	1.691	2.335	1.972	1.713	2.378	1.972	1.713	2.378	1.972	2.6%	2.4%	**5.0%**

$$\boxtimes_{X,Z} = \otimes_{X,Z} - \otimes_\Sigma = 5.0\% - 0.0\%$$

Naïve Bayes

$$\hat{\aleph}_g = \bar{\bar{\aleph}}_\Sigma + \sum_{i=X,Y,Z} \left(\bar{\bar{\aleph}}_i - \bar{\bar{\aleph}}_\Sigma\right)$$

		Total			Fail %			Estimate %			Power		
		Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot
Inputs	X1	5 000	3 750	8 750	3.53%	21.05%	11.04%	**7.48%**	**15.30%**	**10.83%**	0.14%	-0.03%	10.38%
	X2	3 000	2 250	5 250	8.14%	10.87%	9.31%	**6.53%**	**13.49%**	**9.51%**	0.23%	0.18%	-0.39%
	X3	1 800	1 350	3 150	16.67%	5.45%	11.86%	**8.38%**	**16.97%**	**12.06%**	-0.04%	0.20%	-12.09%
	Tot	9 800	7 350	17 150	7.36%	15.07%	10.66%	**7.35%**	**15.06%**	**10.66%**	0.82%	-0.24%	3.31%
		Naive Obs			Naïve Est			Estimate			Inaccuracy		
Log likelihd	X1	0.153	0.515	0.347	0.167	0.526	0.347	**0.166**	**0.526**	**0.321**	2.8%	2.4%	0.0%
	X2	0.282	0.344	0.310	0.284	0.347	0.310	**0.284**	**0.347**	**0.311**	0.4%	0.6%	0.0%
	X3	0.451	0.212	0.364	0.486	0.272	0.364	**0.486**	**0.272**	**0.394**	7.4%	13.0%	0.0%
	Tot	0.263	0.424	0.339	0.263	0.424	0.339	**0.261**	**0.425**	**0.331**	0.0%	0.0%	0.0%
		Naive Obs			Naïve Est			Estimate			Interaction		
Deviance	X1	1.357	2.799	2.003	1.395	2.865	2.003	1.395	2.866	1.899	0.8%	0.5%	1.3%
	X2	1.759	1.989	1.858	1.766	2.002	1.858	1.764	2.000	1.861	0.1%	0.1%	0.2%
	X3	2.462	1.527	2.071	2.644	1.724	2.071	2.645	1.723	2.201	0.8%	1.0%	1.8%
	Tot	1.691	2.335	1.972	1.691	2.335	1.972	1.686	2.338	1.939	1.7%	1.6%	**3.3%**

Single Naïve X/Z interaction term

$$\widehat{\aleph}_g = \bar{\bar{\aleph}}_{X,Z}$$

		Total			Fail %			Estimate %		
		Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot
Inputs	X1	5 000	3 750	8 750	3.53%	21.05%	11.04%	**3.53%**	**21.05%**	**11.04%**
	X2	3 000	2 250	5 250	8.14%	10.87%	9.31%	**8.14%**	**10.87%**	**9.31%**
	X3	1 800	1 350	3 150	16.67%	5.45%	11.86%	**16.67%**	**5.45%**	**11.86%**
	Tot	9 800	7 350	17 150	7.36%	15.07%	10.66%	**7.36%**	**15.07%**	**10.66%**
		Naive Obs			Naïve Est			Estimate		
Log likelihd	X1	0.153	0.515	0.347	0.153	0.515	0.347	**0.153**	**0.515**	**0.308**
	X2	0.282	0.344	0.310	0.282	0.344	0.310	**0.282**	**0.344**	**0.309**
	X3	0.451	0.212	0.364	0.451	0.212	0.364	**0.451**	**0.212**	**0.348**
	Tot	0.263	0.424	0.339	0.263	0.424	0.339	**0.247**	**0.407**	**0.316**
		Naive Obs			Naïve Est			Estimate		
Deviance	X1	1.357	2.799	2.003	1.357	2.799	2.003	1.357	2.799	1.851
	X2	1.759	1.989	1.858	1.759	1.989	1.858	1.759	1.989	1.854
	X3	2.462	1.527	2.071	2.462	1.527	2.071	2.462	1.527	2.006
	Tot	1.691	2.335	1.972	1.691	2.335	1.972	1.639	2.256	1.880

RESULTS

Power		
Z1	Z2	Tot
0.00%	0.00%	15.17%
0.00%	0.00%	0.46%
0.00%	0.00%	6.09%
7.54%	5.93%	9.47%
Inaccuracy		
0.0%	0.0%	0.0%
0.0%	0.0%	0.0%
0.0%	0.0%	0.0%
0.0%	0.0%	0.0%
Interaction		
0.0%	0.0%	0.0%
0.0%	0.0%	0.0%
0.0%	0.0%	0.0%
0.0%	0.0%	**0.0%**

Regressed using WOEs of interaction terms

$$\widehat{\aleph}_g = \overline{\overline{\aleph}}_\Sigma + .5472\mathcal{W}_{X,Y} + .9542\mathcal{W}_{X,Z} + .0809\mathcal{W}_{Y,Z}$$

		Total			Fail %			Estimate %			Power		
		Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot
Inputs	X1	5 000	3 750	8 750	3.53%	21.05%	11.04%	**3.62%**	**21.43%**	**11.25%**	0.18%	0.00%	15.21%
	X2	3 000	2 250	5 250	8.14%	10.87%	9.31%	**7.48%**	**10.28%**	**8.68%**	0.49%	0.27%	0.85%
	X3	1 800	1 350	3 150	16.67%	5.45%	11.86%	**16.93%**	**6.11%**	**12.30%**	-0.07%	0.07%	6.03%
	Tot	9 800	7 350	17 150	7.36%	15.07%	10.66%	**7.25%**	**15.20%**	**10.66%**	7.68%	5.96%	9.55%
Log likelihd		Naive Obs			Naïve Est			Estimate			Inaccuracy		
	X1	0.153	0.515	0.347	0.153	0.515	0.347	**0.153**	**0.515**	**0.308**	0.0%	0.0%	0.0%
	X2	0.282	0.344	0.310	0.283	0.344	0.310	**0.282**	**0.343**	**0.308**	0.1%	0.0%	0.0%
	X3	0.451	0.212	0.364	0.451	0.212	0.364	**0.451**	**0.212**	**0.348**	0.0%	0.1%	0.0%
	Tot	0.263	0.424	0.339	0.263	0.424	0.339	**0.247**	**0.407**	**0.315**	0.0%	0.0%	0.0%
Deviance		Naive Obs			Naïve Est			Estimate			Interaction		
	X1	1.357	2.799	2.003	1.357	2.799	2.003	1.357	2.799	1.851	0.0%	0.0%	0.0%
	X2	1.759	1.989	1.858	1.760	1.990	1.859	1.756	1.987	1.852	0.0%	0.0%	0.0%
	X3	2.462	1.527	2.071	2.462	1.528	2.072	2.463	1.527	2.007	0.0%	0.0%	0.0%
	Tot	1.691	2.335	1.972	1.691	2.335	1.972	1.638	2.255	1.879	0.0%	0.0%	0.0%

Balanced

Total is maintained, known rate is 50 percent

i	1	2	3	4	5	6	7	8	9	Total	Average
w	10.5	5.3	5.3	5.3	10.5	10.5	5.3	5.3	5.3	63.0	
"F"	10.5	0.0	0.0	0.0	10.5	10.5	0.0	0.0	0.0	31.5	50.000%
p Est	73.2%	29.0%	38.2%	63.5%	77.9%	73.2%	51.3%	38.2%	60.1%		
n Est	7.68	1.52	2.01	3.33	8.18	7.68	2.69	2.01	3.15	38.26	60.734%
ll	0.3124	0.3423	0.4815	1.0076	0.2495	0.3124	0.7191	0.4815	0.9184		
*w * ll*	3.2799	1.7970	2.5281	5.2901	2.6193	3.2799	3.7754	2.5281	4.8218	29.9196	0.4749

Label	Rate	Odds	Log-L'hood	Like-lihood	Devi-ance	Inaccuracy
Naive Obs	50.000%	1.000	0.6931	2.0000	4.0000	4.831%
Naive Est	60.734%	0.647	0.7167	2.0477	4.1932	**Power**
Test Est			0.4749	1.6079	2.5853	50.356%

Up from 0.019% and 28.960% respectively

Naïve model revisited

		Total			Fail %			Estimate %			Power		
		Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot	Z1	Z2	Tot
Inputs	X1	3 528	5 359	8 887	23.48%	69.08%	50.98%	**50.00%**	**50.00%**	**50.00%**	0.00%	0.00%	0.00%
	X2	2 688	2 270	4 958	42.62%	50.55%	46.25%	**50.00%**	**50.00%**	**50.00%**	0.00%	0.00%	0.00%
	X3	2 246	1 059	3 306	62.63%	32.56%	52.99%	**50.00%**	**50.00%**	**50.00%**	0.00%	0.00%	0.00%
	Tot	8 462	8 688	17 150	39.95%	59.79%	50.00%	**50.00%**	**50.00%**	**50.00%**	0.00%	0.00%	0.00%
		Naive Obs			Naïve Est			Estimate			Inaccuracy		
Log likelihd	X1	0.545	0.618	0.693	0.693	0.693	0.693	**0.693**	**0.693**	**0.693**	34.5%	16.1%	0.0%
	X2	0.682	0.693	0.690	0.693	0.693	0.693	**0.693**	**0.693**	**0.693**	2.2%	0.0%	0.6%
	X3	0.661	0.631	0.691	0.693	0.693	0.693	**0.693**	**0.693**	**0.693**	6.7%	13.2%	0.4%
	Tot	0.673	0.674	0.693	0.693	0.693	0.693	**0.693**	**0.693**	**0.693**	4.1%	3.9%	0.0%
		Naive Obs			Naïve Est			Estimate			Interaction		
Deviance	X1	2.974	3.445	3.998	4.000	4.000	4.000	4.000	4.000	4.000	7.1%	5.0%	12.1%
	X2	3.914	4.000	3.978	4.000	4.000	4.000	4.000	4.000	4.000	0.3%	0.0%	0.3%
	X3	3.750	3.533	3.986	4.000	4.000	4.000	4.000	4.000	4.000	0.9%	0.8%	1.7%
	Tot	3.841	3.849	4.000	4.000	4.000	4.000	4.000	4.000	4.000	8.3%	5.9%	**14.2%**

$$\boxtimes_T = \sum_{i=1}^{|A|}\sum_{j=1}^{|B|}\left(4 - \bar{\bar{\mathfrak{D}}}_{i,j}\right)/\bar{\bar{\mathfrak{D}}}_{i,j} \times \mathfrak{f}_{i,j}$$

Can apply without estimates

Was 5.0%